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Amalgamation of Matroids and Its Applications

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INTRODUCTION

The Ramsey-type statement has the following form: Given objects A, B (of a structure) there exists an object C with the following property:

For every partition into two classes, of the set of subobjects of C which are isomorphic to A there exists a subobject B' of C isomorphic to B such that all subobjects of B' which are isomorphic to A belong to one class of the partition.

There does not exist a sufficiently general and powerful theory (of structures) which would give a possibility of uniform proofs for concrete (valid) instances of Ramsay-type statements. However, most known Ramsey-type statements are related to partitions of submatroids (of a given type) of a matroid. We give just a few examples:

(i) Ramsey's theorem [15] deals with partitions of submatroids of homogeneous matroids;

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(ii) the vectorspace analogue of the Ramsey theorem [6] deals with partitions of flats of vectorspace matroids;

(iii) the Galvin–Ramsey property [11] deals with partitions of homogeneous submatroids of a graphical matroid.

This paper is an attempt to prove a Ramsey-type statement which deals with matroids directly. We prove

THEOREM. *Let $M(X)$ be a matroid without loops. Then there exists a matroid $N(Y)$ such that for every partition $Y = Y_1 \cup Y_2$ there exists a submatroid $M'(X')$ of $N(Y)$ such that $M' \cong M$ and $X' \subseteq Y_i$ for either $i = 1$ or $i = 2$.*

Our method is similar to [10], but we also use results of [13, 14]. It also enables us to prove a strong necessary condition for the validity of a Ramsey-type theorem for matroids (roughly: one cannot partition non-homogeneous matroids, see Section 3.2 below).

The difficulties are caused by the fact that the amalgamation ("glueing") of matroids is not an easy operation (see, e.g., [7–9]). It was studied under different names (such as simultaneous extensions and symmetric powers).

In Section 1 we prove that matroids can be amalgamated with respect to a hypergraph without short cycles and in a sense this construction is the best possible, see Section 3.1 below.

In Section 2 we give several applications while in Section 3 we prove several necessary conditions for a Ramsey matroid theorem.

1. Amalgamation with Respect to a Hypergraph

First, let us recall some hypergraph notions which will be needed below.

A hypergraph H is determined by the set of its vertices V and the system $\mathscr{E}$ of non-void subsets of V. Elements of $\mathscr{E}$ are called edges. We also write $V = V(H)$ and $\mathscr{E} = \mathscr{E}(H)$. We always assume $\bigcup \mathscr{E} = V$ and hence the set of vertices is sometimes omitted.

If every edge of H has k elements then H is called a k-hypergraph.

A sequence $v_0, E_1, v_1, \ldots, v_{t-1}, E_t, v_t$ is called a path of length t in H if $E_1, \ldots, E_t$ are distinct edges of H, $v_0, v_1, \ldots, v_t$ are distinct vertices of H and $v_{i-1} \in E_i$, $v_i \in E_i$, $i = 1, \ldots, t$. If $t > 1$ and $v_t = v_0$ then the path is called a cycle of length t.

H is called a forest-hypergraph in the case that H does not contain cycles.

Clearly, H does not contain cycles of length 2 if and only if $|E \cap E'| \leqslant 1$ for any distinct edges of H (such hypergraphs are sometimes called simple). It is easy to see that H does not contain cycles of lengths $2, 3, \ldots, k$ iff any k edges of H form a forest.

Let $H = (V, \mathscr{E})$ be a hypergraph, A a fixed set of vertices of H. Denote by H_A the subhypergraph [1] of H induced by the set A; explicitly $H_A = (A, \mathscr{E}')$, where $\mathscr{E}' = \{A \cap E; E \in \mathscr{E}, A \cap E \neq \emptyset\}$. The set A is called connected if the hypergraph H_A is connected. A component of A is the set of vertices of a component of H_A. The number of components of A will be denoted by $c(A)$. The length of the longest path in H_A is denoted by diam A.

Concerning matroids we use standard notations (see [3, 8]). A matroid M on the set X will be denoted by $M(X)$. If X' is a subset of X then $M|X'$ denotes the restriction (the submatroid) of M to the set X'. Recall that A is a flat of the matroid $M|X'$ iff $A = B \cap X'$ for a flat B of M.

The following is the principal construction used in this paper:

DEFINITION 1.1. Let $H = (V, \mathscr{E})$, $\mathscr{E} = (E_1, ..., E_n)$, be a hypergraph. Let $\mathscr{M} = (M_1, ..., M_n)$ be a system of matroids, $M_i = M(E_i)$, $i = 1, ..., n$. An *amalgamation of matroids $\mathscr{M}$ by the hypergraph H* is any matroid $M' = M(V)$ which satisfies $M'|E_i = M_i$ for every $i = 1, ..., n$. An amalgamation of matroids $\mathscr{M}$ by the hypergraph H will be denoted by $H * \mathscr{M}$. If $M \simeq M_1 \simeq \cdots \simeq M_n$ then the above amalgamation will be denoted by $H * M$.

For a particular choice of H and M an amalgamation $H * M$ need not exist (this will be shown below). The main result concerns local properties of H which imply the existence of $H * \mathscr{M}$.

In the rest of this section we will assume that no matroids contain loops and parallel vertices. As will be shown in Section 2, this is a pure technical assumption. Consequently, a matroid will be considered as a pair $(X, \mathscr{F})$, where $\mathscr{F}$ is a collection of flats with properties:

(M1) $\{x\} \in \mathscr{F}$ for every $x \in X$;

(M2) for every $A \in \mathscr{F}$, $x \notin A$, there exists a unique $B \supseteq A \cup \{x\}$, $r(A) + 1 = r(B)$.

THEOREM 1.2. *Let $H = (V, \mathscr{E})$ be a hypergraph, let k be the minimal length of a cycle in H (we put $k = \infty$ if H is a forest). Let for every edge $E \in \mathscr{E}$ be given a matroid $M_E = M(E)$ of rank $\leqslant k$. Put $\mathscr{M} = (M_E; E \in \mathscr{E})$. Then there exists an amalgamation $H * \mathscr{M}$ of rank*

$$\min \left(k, c(V) - |\mathscr{E}| + \sum_{E \in \mathscr{E}} r(M_E)\right).$$

COROLLARY 1.3. *In particular, if M is a matroid of rank k with m vertices and if H is an m-hypergraph without cycles of length $<k$ then there exists a matroid $H * M$.*

Proof of the theorem. Let the hypergraph $H=(V,\mathscr{E})$ and matroids $\mathscr{M}=(M_E;E\in\mathscr{E})$ satisfy the promises of Theorem 1.2. We say that a subset A of V is *locally closed* if $A\cap E$ is closed in M_E for every $E\in\mathscr{E}$. Observe that the intersection of a family of locally closed sets is locally closed. For a subset A of V put

$$\bar{A}=\bigcap\{B;B\supseteq A, B \text{ locally closed}\}.$$

The following is easy to prove.

(C1) $\bar{A}$ is a locally closed set.

(C2) Define the sequence $A=A_1\subseteq A_2\subseteq\cdots\subseteq A_t\subseteq\cdots$ as follows: If A_t fails to be locally closed then $A\cap E$ fails to be closed in M_E for some $E\in\mathscr{E}$. Put $A_{t+1}=A_t\cup\overline{E\cap A}^E$. Obviously $\bar{A}=A_t$ for some t.

(C3) Obviously $\bar{A}=\bar{A}^E$ for $A\subseteq E$. We also write $r(A)$ for the rank of the set $A\subseteq E$ in the matroid M_E.

The matroid structure of $H*\mathscr{M}$ will be induced by means of the function

$$f(A)=\sum(r(E\cap A)-1;E\cap A\neq\varnothing)+c(A).$$

The function f has the following properties (for every $A\subseteq V$):

(P0) $f(A)=\sum(f(B);B$ a component of $A)$;

(P1) $\operatorname{diam}A<f(A)$;

(P2) if A is a locally closed set and $A\subsetneq B$ then $f(A)<f(B)$;

(P3) $f(A\cup\{x\})\leqslant f(A)+1$ providing that $f(A)<k-1$;

(P4) $f(\bar{A})\leqslant f(A)$ whenever $f(A)<k$.

Proof of these statements will follow the proof of Theorem 1.2.

Define a set $\mathscr{F}$ of subsets of V as follows: $A\in\mathscr{F}$ iff A is locally closed and either $f(A)<k$ or $A=V$.

We prove that $(V,\mathscr{F})$ is a matroid. We also prove that $A\in\mathscr{F}$, $A\neq V$, has rank $f(A)$ (but f fails to be the rank function of $(V,\mathscr{F})$).

1. $\{x\}\in\mathscr{F}$ for every $x\in V$ (as $f(x)=1$);

2. let $A\in\mathscr{F}$, $x\notin A$, be given. If $f(A)=k-1$, then V is the unique flat containing both A and x. Let $f(A)<k-1$. Then there holds

$$f(A)<f(\overline{A\cup\{x\}})\leqslant f(A\cup\{x\})\leqslant f(A)+1.$$

Here, the first inequality follows from (P2), the second one from (P4) and the last one from (P3). Consequently

$$f(\overline{A \cup \{x\}}) = f(A) + 1.$$

The unicity of $B = A \cup \{x\}$ follows from (P2).

Let $E \in \mathscr{E}$. It remains to be proved that $(V, \mathscr{F}) \mid E = M_E$. According to (C3) above $A \subsetneq E$ is closed in M_E iff $A \in \mathscr{F}$. This finishes the proof of Theorem 1.2.

Proof of properties (P0)–(P4). Let the hypergraph $H = (V, \mathscr{E})$ and matroids M_E, $E \in \mathscr{E}$, be as above.

(P0) Let $A_1, \dots, A_p$ be all the components of A. Put

$$\mathscr{E}' = \{E \in \mathscr{E}; E \cap A \neq \varnothing\}$$

and

$$\mathscr{E}_i = \{E \in \mathscr{E}; E \cap A_i \neq \varnothing\}.$$

Clearly $(\mathscr{E}_1, \mathscr{E}_2, \dots, \mathscr{E}_p)$ is a partition of $\mathscr{E}'$ and

$$f(A) = \left(\sum_{E \in \mathscr{E}'} (r(A \cap E) - 1) + 1 \right)$$

$$= \sum_{i=1}^{p} \left(\sum_{E \in \mathscr{E}_i} (r(A \cap E) - 1) + 1 \right) = \sum_{i=1}^{p} f(A_i).$$

(P1) Let $v_0, E_1 \cap A, v_1, \dots, v_{t-1}, E_t \cap A, v_t$ be the longest path in H_A. Then

$$f(A) = \sum (r(A \cap E) - 1; A \cap E \neq \varnothing) + c(A)$$

$$\geqslant (r(A \cap E_i) - 1) + 1 \geqslant t + 1,$$

as $|A \cap E_i| \geqslant 2$ and the matroids M_{E_i} do not contain parallel vertices.

(P2) Let $A \subsetneq B$, A locally closed, be fixed. Using (P0) we may assume that B is connected. Let A_i, $i = 1, \dots, p$, be all the components of A. If $p = 1$ then $r(A \cap E) < r(B \cap E)$ for some $E \in \mathscr{E}$ as $A \cap E$ is closed in M_E. Assume $p > 1$. Fix a component A_i and vertices x, y, $x \in A_i$, $y \in B - A_i$. By the connectivity of B there exists a path

$$x = x_0, E_1 \cap B, x_1, \dots, x_{t-1}, E_t \cap B, x_t = y.$$

This proves that there exists an edge $E \in \mathscr{E}$ such that

$$|B \cap E| > |A_i \cap E| > 0. \tag{1}$$

For every $i = 1,\ldots,p$ let an edge $E_i \in \mathscr{E}$ with (1) be fixed. As A is locally closed it follows that

$$r(B \cap E_i) > r(A_i \cap E_i) > 0. \tag{2}$$

Observe, that $E_i \neq E_j$ for $i \neq j$ (for otherwise $A_i \cup A_j$ would be connected). Summarizing these observations (and (P0)) we get

$$\sum_{E \in \mathscr{E}_i} (r(A \cap E) - 1) + 1 \leqslant \sum_{E \in \mathscr{E}_i} (r(B \cap E) - 1)$$

and consequently

$$\begin{aligned} f(A) &= \sum_{i=1}^{p} \sum_{E \in \mathscr{E}_i} (r(A \cap E) - 1) + p \\ &= \sum_{i=1}^{p} \left(\sum_{E \in \mathscr{E}_i} (r(A \cap E) - 1) + 1 \right) \\ &< \sum_{i=1}^{p} \sum_{E \in \mathscr{E}_i} (r(B \cap E) - 1) + 1 \leqslant f(B). \end{aligned}$$

(P3) First, we prove that for every component A_i of A there exists at most one $E \in \mathscr{E}$ such that $x \in E$ and $E \cap A_i \neq \varnothing$. Suppose that there are two edges E, E' such that $x \in E \cap E'$ and $E \cap A_i \neq \varnothing \neq E' \cap A_i$. Fix $y \in E \cap A_i$, $y' \in E' \cap A_i$. There exists a path

$$y = v_0, E_1 \cap A_i, v_1,\ldots, v_{t-1}, E_t \cap A_i, v_t = y'$$

with $t < k - 2$ (as $\operatorname{diam} A_i < f(A) < k - 1$ by (P1)). But then

$$x, E, v_0, E_1, v_1,\ldots, v_t, E', x$$

contains a cycle of length $<k$ in $H = (V, \mathscr{E})$, a contradiction. It follows

$$c(A \cup \{x\}) = c(A) - |\{E; x \in E, E \cap A \neq \varnothing\}| + 1.$$

Consequently

$$\begin{aligned} f(A \cup \{x\}) &= \sum (r((A \cup \{x\}) \cap E) - 1; E \cap (A \cup \{x\}) \neq \varnothing) \\ &\quad + c(A \cup \{x\}) \\ &\leqslant \sum (r(A \cap E) - 1; A \cap E \neq \varnothing) \\ &\quad + |\{E; x \in E, A \cap E \neq \varnothing\}| + c(A \cup \{x\}) \\ &= f(A) + 1. \end{aligned}$$

(The inequality follows from $r((A \cup \{x\}) \cap E) \leqslant r(A \cap E) + 1$.)

(P4) According to the above remark (C3) it suffices to show the validity of the following statement:

$f(A \cup \{x\}) \leqslant f(A)$ for every set $A \subseteq V$ and $x \notin A$ such that x belongs to the closure of $A \cap E_1$ for some $E_1 \in \mathscr{E}$.

Let A, x, E_1 be fixed. Using (P0), $A \cup \{x\}$ may be assumed to be connected. Let $A_1, \ldots, A_p$ be all components of A, assume without loss of generality that $A \cap E_1 \subseteq A_1$. We prove

$$f(A_1 \cup \{x\}) \leqslant f(A_1).$$

Clearly, it suffices to prove

$$r(E \cap (A_1 \cup \{x\})) \leqslant r(E \cap A_1)$$

for every edge $E \in \mathscr{E}$, $E \cap A_1 \neq \varnothing$. Assume $r(E \cap (A \cup \{x\})) > r(E \cap A)$ for some $E \in \mathscr{E}$, $E \cap A_1 \neq \varnothing$. It is $|E_1 \cap A_1| \geqslant 2$ (as x belongs to the closure of $E_1 \cap A_1$) and therefore there exists a path

$$x_0, E_1 \cap A_1, x_1, \ldots, x_t$$

in H_{A_1} such that $x_t \in E \cap A_1$. It follows from (P1) that $t < k - 1$. But then $x, E_1, x_1, \ldots, x_t, E, x$ contains a cycle of length $<k$ in H, a contradiction. Thus

$$f(A_1 \cup \{x\}) \leqslant f(A_1).$$

Moreover,

$$f(A_i \cup \{x\}) \leqslant f(A_i) + 1$$

for every $i = 2, \ldots, p$ by (P3) (as $f(A_i) < k - 1$).

Summarizing these observations we get

$$\begin{aligned} f(A \cup \{x\}) &= \sum_{i=1}^{p} f(A_i \cup \{x\}) - p + 1 \\ &= f(A_1 \cup \{x\}) + \sum_{i=2}^{p} f(A_i \cup \{x\}) - p + 1 \\ &\leqslant f(A_1) + \sum_{i=2}^{p} (f(A_i) + 1) - p + 1 \\ &= \sum_{i=1}^{p} f(A_i) = f(A). \end{aligned}$$

This proves (P4).

Remarks. The construction given in the proof of Theorem 1.2 generalizes some standard constructions of matroids.

1. If $H=(V, \mathscr{E})$ and $(E_1, E_2, \ldots, E_n)$ is a partition of V then H does not contain any cycle and $H * \mathscr{M}$ is the sum of matroids $(M_E; E \in \mathscr{E})$.

2. If $H=(V, \mathscr{E})$ and $\mathscr{E}=\{E_1, E_2\}$, $|E_1 \cap E_2|=1$, then H does not contain cycles and $H * \mathscr{M}$ is the Brylawski's one point join (see [2]) of M_{E_1} and M_{E_2}.

3. If $H=(V, \mathscr{E})$ is a forest-hypergraph then an amalgamation $H * \mathscr{M}$ may be constructed by an induction using matroid sums and Brylawski's one point joins. It may be proved that the resulting matroid is uniquely determined by H and $\mathscr{M}$, and that this matroid coincides with the matroid constructed in the proof of Theorem 1.2.

2. Applications

It is known that there are sparse hypergraphs (i.e., hypergraphs without short cycles) which are globally very dense (i.e., which have a large chromatic number), see [4]. This is related to partition Ramsey theory, see [10, 14]. Therefore it is not surprising that Theorem 1.2 may be applied to yield a Ramsey-type result for matroids. In this part we give this and related applications. In fact these results provided the original motivation for our research.

First, we state some definitions from partition theory (see [12] for related notions for graphs and hypergraphs):

Definitions 2.1. Let $M=M(X)$, $N=N(Y)$ be matroids, $r \geqslant 1$ a natural number. We write $M \to_r^1 N$ if for any mapping $c\colon Y \to \{1, \ldots, r\}$ there exists a submatroid $M'=M'(X')$ of N such that $M' \simeq M$ and $c|_{X'}$ is a constant mapping. We say that N has *Folkman property* for M.

We write $M \to_{\mathrm{sel}} N$ if for every mapping $c\colon Y \to Y$ there exists a submatroid $M'=M'(X')$ of N such that $M' \simeq M$ and $c|_{X'}$ is either 1–1 or a constant. In this case we say that N has the *selective property* for M.

We write $M \to_{\mathrm{ord}} N$ if for every total ordering $\leqslant$ of X and for every total ordering $\preccurlyeq$ of Y there exists a submatroid $M'=M'(X')$ of N such that there exists an isomorphism $\varphi\colon M \to M'$ which is a monotone mapping with respect to $\leqslant$ and $\preccurlyeq|_{X'}$. In this case we say that N has the *ordering property* for M.

Sometimes we write $(M, \leqslant) \to_{\mathrm{ord}} N$ which means the following: For every (total) ordering $\preccurlyeq$ of Y there exists a submatroid $M'=M'(X')$ of N such that there exists an isomorphism $\varphi\colon M \to M'$ which is a monotone mapping with respect to $\leqslant$ and $\preccurlyeq|_{X'}$.

It is easy to see that if for every $(M, \leqslant)$ there exists an N such that $(M, \leqslant) \to_{\text{ord}} N$ then for every M there exists an N such that $M \to_{\text{ord}} N$.

We prove

THEOREM 2.2. (a) *For a matroid M and an integer $r \geqslant 2$ there exists a matroid N such that $M \to_r^1 N$ iff either M is loopless or M has rank* 0.

(b) *For a matroid M there exists a matroid with the selective property iff either M is loopless or M has rank* 0 *or has at most two vertices.*

(c) *For a matroid M there exists a matroid with the ordering property iff either M is loopless or M has rank* 0.

Proof. Proofs of (a), (b), (c) follow the same pattern. The easier part is the necessity:

If M contains both loops and independent vertices then the mapping $c: Y \to \{1, 2\}$ defined by $c(y) = 1$ if y is a loop, $c(y) = 2$ otherwise shows the impossibility of both $M \to_2^1 N(Y)$ and $M \to_{\text{sel}} N(Y)$. Moreover, every ordering $\leqslant$ of X satisfying $x < x'$ for all x loop and x' independent and every ordering $\preccurlyeq$ of Y satisfying $y \prec y'$ for all y independent and y' loop show the impossibility of $M(X) \to_{\text{ord}} N(Y)$.

Consequently, if M contains loops then $r(M) = 0$. Furthermore if $r(M) = 0$ then $M \to_{\text{ord}} M$ and the existence of an N with $M \to_r^1 N$ and $M \to_{\text{sel}} N$ is equivalent to suitable variants of Dirichlet's principle.

It remains to consider M loopless.

Let $M^\vee = M/\sim$ be a factorization of M by the equivalence $x \sim y$ iff $x = y$ or x and y are parallel. Clearly $M^\vee = M^\vee(X^\vee)$ does not contain loops and parallel vertices.

Put $|X^\vee| = m$, $r(M^\vee) = r(M) = k$. Let H be a hypergraph with the following properties:

(1) H is an m-hypergraph,

(2) H does not contain cycles of length $<k$,

(3a) $\chi(H) > r$ (χ denotes the chromatic number of H),

(3b) H has the selective property (see [13], this means the following: for every mapping $c: V(H) \to V(H)$ there exists an edge $E \in \mathscr{E}(H)$ such that $c|_E$ is either a constant or a 1–1 mapping),

(3c) there exists a fixed ordering $\leqslant_E$ of every edge of H (orderings $\leqslant_E$ and $\leqslant_{E'}$ are independent for $E \neq E'$) such that for every ordering $\preccurlyeq$ of $V(H)$ there exists an edge $E \in \mathscr{E}(H)$ such that $\preccurlyeq|_E = \leqslant_E$.

A hypergraph H with properties (1), (2) and (3a) exists by [4], a hypergraph H with properties (1), (2) and (3b) exists by [13] and a hypergraph H with properties (1), (2) and (3c) exists by [14].

Let us remark that the existence of hypergraphs used in (b) and (c) uses probabilistic means and consequently presently the proof of Theorem 2.2, parts (b) and (c), has a nonconstructive character.

By Corollary 1.3 we get $N^{\vee} = H * M^{\vee}$. It is easy to see that $M^{\vee} \to_r^1 N^{\vee}$ and $M^{\vee} \to_{\text{sel}} N^{\vee}$.

The ordering property needs a bit more care. In order to get $(M^{\vee}, \leqslant) \to_{\text{ord}} N$ choose the amalgamation $N^{\vee} = H * M^{\vee}$ so that $N^{\vee}$ restricted to any edge $E \in \mathscr{E}(H)$ is isomorphic to M and this isomorphism is the monotone mapping (with respect to $\leqslant$ and $\leqslant_E$; this is clearly possible by the construction of $H * M$).

It remains to add to N a suitable number of parallel vertices. To do so put $a = |X| - |X^{\vee}| + 1$ and $b = \max\{r(a-1)+1, a(a-1)+1\}$. Let N be the matroid which arises from $N^{\vee}$ by replacing every vertex of $N^{\vee}$ by b parallel vertices. By an easy combination of Dirichlet's principle and the above properties of $N^{\vee}$ we get

$$M \xrightarrow[r]{1} N, \qquad M \xrightarrow[\text{sel}]{} N, \qquad (M, \leqslant) \xrightarrow[\text{ord}]{} N$$

(actually, to prove $(M, \leqslant) \to_{\text{ord}} N$ it suffices to put $b = a$).

3. Concluding Remarks

1. *Amalgamation with Respect to General Hypergraph*

Theorem 1.2 is in a sense the best possible. Figure 1 gives an example of a hypergraph H without cycles of length <3 and if a family $\mathscr{M} = (M_1, M_2, \ldots, M_6)$ of matroids of rank $\leqslant 4$ satisfies $r(M_1) = \cdots = r(M_5) = 2$, $r(M_6) = 4$ then $H * \mathscr{M}$ does not exist ($M_1, M_2, \ldots, M_5$ are coplanar lines in any amalgamation $H * \mathscr{M}$ and hence $r(M_6) \leqslant 3$).

2. *Remark on General Ramsey Matroids*

Denote by $\binom{N}{M}$ the set of all submatroids of N which are isomorphic to M.

Given matroids M, N, P and $r \geqslant 1$ we write $M \to_r^P N$ if the following statement is true:

For every mapping $c: \binom{N}{P} \to \{1, \ldots, r\}$ there exists a submatroid M' of N, $M' \simeq M$, such that c restricted to the set $\binom{M'}{P}$ is a constant mapping.

We say that N is a P-Ramsay matroid for M.

In Section 2 we proved that for every loopless matroid there exists a point-Ramsey matroid. The natural question ("the Ramsey problem") is for which P do P-Ramsey matroids exist.

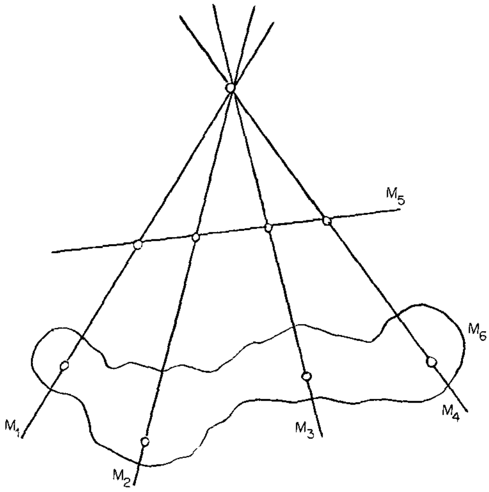

FIGURE 1

Theorem 2.2 implies that this is not always the case:

COROLLARY 3.2.1. *Let P be a fixed matroid. Suppose that for every matroid M and every integer $r \geqslant 1$ there exists a matroid N such that $M \to_r^P N$. Then P is a k-uniform matroid for some k (see [8]).*

Proof. The following is an alternative description of a uniform matroid: $P(Z) = P$ is uniform iff every bijection $f: Z \to Z$ is an isomorphism of $P \to P$.

Consequently if P fails to be a uniform matroid then there are orderings $\leqslant_1$ and $\leqslant_2$ of X such that the monotone bijection $(Z, \leqslant_1) \to (Z, \leqslant_2)$ fails to be an isomorphism $P \to P$. Let $M(X) = M$ be the matroid sum of P and P and consider X with ordering $\leqslant$ which extends both $\leqslant_1$ and $\leqslant_2$. Let $(M, \leqslant) \to_{\text{ord}} M^\vee = M^\vee(X^\vee)$. Then $M^\vee \to_2^P N$ for no N. This can be seen as follows (in a different context this argument appears in [11]):

Put $N = N(Y)$ and fix an (total) ordering $\preccurlyeq$ of Y. Define a mapping $c: \binom{N}{P} \to \{1, 2\}$ as follows:

$c(P') = 1$ if $P' = P'(Z')$ and the monotone (with respect to $\leqslant_1$ and $\preccurlyeq|_{Z'}$) bijection $Z \to Z'$ is an isomorphism $P \to P'$;

$c(P') = 2$ otherwise.

It is easy to see that this mapping disproves $M^\vee \to_2^P N$.

3. *Remark on Flats*

The following stronger form of Theorem 2.2a is valid:

THEOREM 3.3.1. *For every matroid $M(X)$ without loops and every integer $r \geqslant 1$ there exists a matroid $N(Y)$ with the following property:*

For every partition $Y = \bigcup_{i=1}^{r} Y_i$ there exists i and a flat $X' \subseteq Y_i$ such that $N|X'$ is isomorphic to M.

(A proof is provided by the same construction as in Theorem 1.2.) For the sake of brevity put $M \to_r^{1,f} N$ in the case that the statement of Theorem 3.3.1 is true.

This flat-embedding arrow is interesting even in the simplest cases: Denote by L_n the line with n vertices (the uniform matroid of rank 2 with n vertices). Obviously $L_n \to_r^1 L_{r(n-1)+1}$ while for a set X the existence of $M(X)$ with $L_n \to_r^{1,f} M(X)$ is equivalent to the existence of an n-hypergrapg $H = (X, \mathscr{E})$ satisfying

(1) H does not contain cycles of length 2,

(2) $\chi(H) > r$.

The Fano matroid is the smallest matroid for which $L_3 \to_2^{1,f} M$. (The asymptotic behaviour of $|X|$ for hypergraphs $(X, \mathscr{E})$ with (1), (2) is determined in [5].)

4. *Remark on General Ramsey Matroids* II

The necessary condition given above in Section 3.2 is not a sufficient condition for the existence of P-Ramsey matroids. To see this, consider matroids M and M' depicted on Fig. 2 (M, M' have rank 3 and only the non-trivial lines are indicated).

It can be verified that there does not exist an amalgamation of M and M' such that the sets A and A' are identified. However, the restrictions $M|A$ and $M'|A'$ are uniform matroids which are isomorphic. Put $P = M|A \simeq M'|A'$.

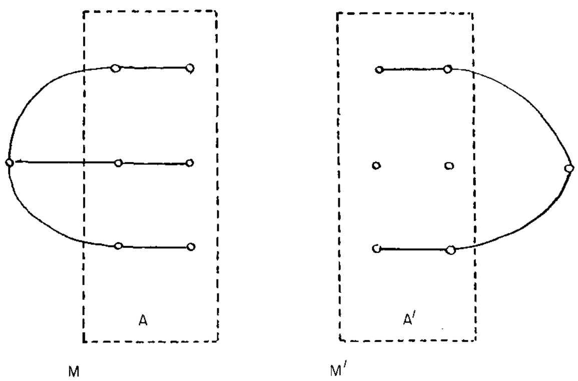

FIGURE 2

Let $M^{\vee}$ be the matroid sum of M and M' (we use the notation $M^{\vee} = M + M'$). We prove that there does not exist a matroid N such that $M^{\vee} \to_2^P N$.

Outline of proof. Observe $|\binom{M^{\vee}}{P}| = 2$. Define the directed graph G as follows:

$$V(G) = \binom{N}{P},$$

$$(P_1, P_2) \in E(G) \quad \text{iff} \quad P_1 \in \bar{M},\ P_2 \in \bar{M}' \text{ for some } \bar{M} + \bar{M}' \in \binom{N}{M^{\vee}}.$$

Obviously $M^{\vee} \to_2^P N$ implies $\chi(G) > 2$. Consequently there are edges (P_1, P_2) and (P_1', P_2') such that $P_2 = P_1'$. Using the definition of $M^{\vee}$, this in turn means that a convenient restriction of N is an amalgamation of M and M' with respect to P.

We have some evidence that P-Ramsey matroids exist for some particular homogeneous P. Particularly, we suspect that the following is true:

Conjecture 3.4. Let L be the discrete matroid with 2 vertices. Then for every matroid M without loops there exists an L-Ramsey matroid. Explicitly: For every matroid M without loops there exists a matroid N such that for every red–blue partition of 2-lines in N there exists a copy M' of M in N all whose 2-lines are either blue or red.

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